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A Note on Relaxed Control Functions

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In [2] the relaxed control functions were defined as linear functionals on some Banach space $L_1(I, C(S))$ of integrable functions with domain I and values in the space of real valued continuous functions defined on some compact set S. The purpose of this note is to extend the range of relaxed control functions to every function $\phi: I \times S \to \mathbb{R}$ jointly measurable and bounded.

1. INTRODUCTION AND NOTATION

Let $(I, \mathcal{M})$ be a general measurable space and (S, d) a compact metric space. $\mathcal{B}(S)$ will denote the σ-algebra of Borel subsets of S and $\tau pm(S)$ will denote the set of Radon probability measures defined on $\mathcal{B}(S)$. If μ is a positive measure defined on $\mathcal{M}$, let $\mathcal{M}_\mu$ define the μ-completion of $\mathcal{M}$, that is the σ-algebra generated by $\mathcal{M}$ and all the subsets of all μ-nullsets. Denote by $\tilde{\mathcal{M}}$ the completion of $\mathcal{M}$, that is $\tilde{\mathcal{M}} = \bigcap_\mu \mathcal{M}_\mu$, where μ ranges over all bounded positive measures defined on $\mathcal{M}$. It is evident that $\mathcal{M} \subset \tilde{\mathcal{M}} \subset \mathcal{M}_\mu$ all μ.

Let $P(S)$ denote the set of all subsets of S, a map $A: I \to P(S)$ is called a multifunction. $\operatorname{gph} A = \{(t, s) \in I \times S\colon s \in A(t)\}$ is called the graph of the multifunction A.

Let $L_1(I, C(S))$ denote the space of ν-integrable functions with domain I and values in the space of real functions defined on S, where ν some positive, nonatomic measure on $\mathcal{M}$. $L_1(I, C(S))$ equipped with the norm

$$\|\phi\| = \int_I \|\phi(t, \cdot)\|_{\sup} \, d\nu \qquad \phi \in L_1(I, C(S))$$

is a Banach space ([3, IV.1.2]).

It is a consequence of the Dunford–Pettis theorem that the topological dual of $L_1(I, C(S))$ is isometrically isomorph to the Banach space $L_\infty(I, C^*(S))$ that is the space of ν-essentially bounded measurable functions with domain I and values in the space $C^*(S)$ of Radon measures on $\mathcal{B}(S)$.

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The space of relaxed control functions corresponding to S is the subset $G(S)$ of $L_\infty(I, C^*(S))$ comprising of elements μ with strong norm 1, i.e.,

$$\text{ess sup } \|\mu(t)\|_{\text{T.v}} = 1$$

where $\|\cdot\|_{\text{T.v.}}$ the total variation norm. That is $\mu(t) \in \tau pm(S)$ for v-a.e. t.

$G(S)$ is a convex set and when it is equipped with the relativization of the weak * topology of $L_\infty(I, C^*(S))$ it is a metrisable compact space (Banach–Alocoglu). Recall that the duality is expressed through

$$\mu(\phi) = \int_I dv \int_S \phi(t, s)\, \mu(t)(ds), \qquad \phi \in L_1(I, C(S)), \quad \mu \in G(S). \tag{1}$$

In [2] the range of the linear functional μ is extended to a space that contains $L_1(I, C(S))$, namely to functions jointly measurable and only semicontinuous in s and a lower (upper)-semicontinuity property was established. We shall employ the theory of multifunctions to show that $\mu(\phi)$ in (1) is meaningful for any $\phi \mathscr{M} \otimes \mathscr{B}(s)$ measurable and bounded.

2. Main Result

Lemma 1. *Every $(t, s) \mapsto \phi(t, s)$, $\mathscr{M} \otimes \mathscr{B}(S)$-measurable and bounded below is the limit of some sequence of linear combinations of "characteristic functions" of the form $\chi_{A(t)}(s)$, where $A: I \to P(S)$ is a multifunction with measurable graph.*

Proof. Suppose $A \subset I \times S$ is $\mathscr{M} \otimes \mathscr{B}(S)$ measurable. Take $B(t) = \{s \in S: (t, s) \in A\}$ that is $A = \text{gph} B$. Then

$$\begin{aligned} \chi_A(t, s) &= 1 \Leftrightarrow (t, s) \in A \Leftrightarrow s \in B(t) \Leftrightarrow \chi_{B(t)}(s) = 1 \\ &= 0 \Leftrightarrow (t, s) \notin A \Leftrightarrow s \notin B(t) \Leftrightarrow \chi_{B(t)}(s) = 0 \end{aligned}$$

that is, $\chi_A(t, s) = \chi_{B(t)}(s)$ all $(t, s) \in I \times S$.

Since ϕ is $\mathscr{M} \otimes \mathscr{B}(S)$ measurable and bounded below there exists an increasing sequence $\{\xi_n\}$ of the form $\xi_n(t, s) = \sum_{i=1}^{N_n} a_i \chi_{A_{ni}}(t, s)$ where $A_{ni} \in \mathscr{M} \otimes \mathscr{B}(S)$ $n \in \mathbb{N}$, $a_i \in \mathbb{R}$, $1 \leqslant i \leqslant N_n \in \mathbb{N}$, and $\xi_n \to \phi$ pointwise. But then for each (n, i), $n \in \mathbb{N}$, $1 \leqslant i \leqslant N_n$, there exists $B_{ni}: I \to P(S)$ with $A_{ni} = \text{gph} B_{ni} \in \mathscr{M} \otimes \mathscr{B}(S)$ and such that $\chi_{A_{ni}}(t, s) = \chi_{B_{ni}}(s)$. This proves the assertion.

Lemma 2. *Assume μ is a relaxed control function and $A: I \to P(S)$ has measurable graph. Then*

$$t \mapsto \mu(t)[A(t)]$$

is $\tilde{\mathscr{M}}$-measurable.

Proof. Denote $\mathscr{A}(t)=\{p\in\tau pm(S): p[A(t)]\leqslant a\}$, $a\in\mathbb{R}$. From [1, Theorem IV.12], $t\mapsto\mathscr{A}(t): I\to P(\tau pm(S))$ has $\mathscr{M}\otimes\mathscr{B}(\tau pm(S))$-measurable graph. We then have

$$\begin{aligned}
S_a&=\{t\in I: \mu(t)[A(t)]\leqslant a\}=\{t\in I: \mu(t)\in\mathscr{A}(t)\}\\
&=\{t\in I: \mu(t)\cap\mathscr{A}(t)\neq\varnothing\}\\
&=\mathrm{proj}_I\{(t,s)\in I\times\tau pm(S): s=\mu(t), s\in\mathscr{A}(t)\}\\
&=\mathrm{proj}_I[\mathrm{gph}p\cap\mathrm{gph}\mathscr{A}].
\end{aligned}$$

But $\mathrm{gph}p\cap\mathrm{gph}\,\mathscr{A}\in\mathscr{M}\otimes\mathscr{B}(\tau pm(S))$, hence the projection theorem, [1 Theorem III.22], implies that S_a is $\bar{\mathscr{M}}$-measurable each $a\in\mathbb{R}$ and therefore $t\mapsto\mu(t)[A(t)]$ is $\bar{\mathscr{M}}$-measurable. ∎

PROPOSITION 1. *Let $\mu\in G(S)$ and $\phi: I\times S\to\mathbb{R}\cup\{+\infty\}$ be $\mathscr{M}\otimes\mathscr{B}(S)$-measurable and bounded below. Then the map $t\mapsto\int_S\phi(t,s)\,\mu(t)(ds)$ is $\bar{\mathscr{M}}$-measurable.*

Proof. Since $s\mapsto\phi(t,s)$ is $\mathscr{B}(S)$-measurable the integral makes sense for each $t\in I$. From Lemma 1 $\phi(t,s)=\lim_n\sum_{i=1}^{N_n}a_i\chi_{B_{ni}}(s)$ therefore

$$\begin{aligned}
\int_S\phi(t,s)\,\mu(t)(ds)&=\int_S\lim_n\sum_{i=1}^{N_n}a_i\chi_{B_{ni}(t)}(s)\,\mu(t)(ds)\\
&=\lim_n\sum_{i=1}^{N_n}a_i\int_S\chi_{B_{ni}(t)}(s)\,\mu(t)(ds)\\
&\qquad\text{(monotone convergence theorem)}\\
&=\lim_n\sum_{i=1}^{N_n}a_i\mu(t)[B_{ni}(t)]. \qquad (2)
\end{aligned}$$

From Lemma 2 $t\mapsto\mu(t)[B_{ni}(t)]$ is $\bar{\mathscr{M}}$-measurable and the assertion follows from (2). ∎

COROLLARY. *If everything is as in Proposition 1 and ϕ is bounded the functional*

$$\mu(\phi)=\int_I dt\int_S\phi(t,s)\,\mu(t)(ds)$$

is well defined.

When regarding a control problem, the control constraint $u(t)\in U(t)$ is given through a multifunction $U: I\to P(S)$, following [3] we define the set $G(U)$ of relaxed control functions corresponding to the multifunction U, by

$G(U) = \{\mu \in G(S)\colon \mu(t)[dV(t)] = 1\text{—a.e.}\}$, where d denotes topological closure.

Then $\mu \in G(U)$ retains the measurability properties required if U has measurable graph, and the results hold true. $G(U)$ is a compact and convex subset of $G(S)$.

References

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